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Conformal mechanics and the Virasoro algebra

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Abstract: We demonstrate that any scale-invariant mechanics of one variable exhibits not only 0+1 conformal symmetry, but also the symmetries of a full Virasoro algebra. We discuss the implications for the adS/CFT correspondence.

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Contents

1. Introduction

Recently, there has been much interest in the conjectured correspondence between theories of gravity in anti-de Sitter background and conformal field theories living on the boundary of AdS [1, 2, 3]. A particularly interesting example of this is the AdS_2/CFT_1 correspondence [4, 5, 6]. In particular, Strominger [4] pointed out that quantum gravity on AdS_2 must be a conformal theory on a strip, which exhibits the symmetries of the Virasoro algebra. This suggests that perhaps the boundary conformal theory also exhibits the symmetries of the full Virasoro algebra despite the fact that, naively, the boundary theory is quantum mechanics. This suggestion is lent credence by a recent proposal of Gibbons and Townsend [7], who suggest that the boundary theory for AdS_2 is given by a Calogero model of quantum mechanics. This model is a limit of a more general Calogero model, which was shown to exhibit the symmetries of a Virasoro algebra [8].

In fact the $\mathrm{SO}(1,2)$ isometry group of AdS_2, which forms the 0+1 conformal group, is given by the charge-free subalgebra of the Virasoro algebra generated by L_1, L_0 and L_{-1}. A conformal mechanics describing the dynamics of a particle in the background of a Reisnner-Nordstrom black hole was given in [9]. Motivated by [4], we consider the system given in [9] to determine if one can find generators of the full Virasoro algebra. Not only do we find generators of the full algebra, but we also find (somewhat surprisingly) that given any scale-invariant Hamiltonian of one dynamical variable, one can classically find generators of the full Virasoro algebra (the global subalgebra of which generates 0+1 conformal symmetry).

2. Finding the generators

One can write the 0+1 dimensional conformal algebra in the following manner

$$[D,H]=H\,,\qquad [D,K]=-K\,,\qquad [H,K]=2D\,, \tag{2.1}$$

where H is the Hamiltonian, D is the generator of dilatations, and K is the generator of the special conformal symmetry. By making the identification

$$L_0=-\imath D\,,\qquad L_{-1}=-\imath H\,,\qquad L_1=\imath K \tag{2.2}$$

we see that the L_0, L_{-1} and L_1 satisfy the Virasoro algebra

$$[L_m,L_n]=-\imath(m-n)L_{m+n}+A(m)\delta_{m+n}\,. \tag{2.3}$$

In [9], we find

$$D=\frac{1}{2}pr\,,\qquad H=\frac{p^2}{2f}\,,\qquad K=-\frac{1}{2}fr^2 \tag{2.4}$$

and

$$f=\frac{1}{2}m\left[\sqrt{1+(\frac{rp}{2mM})^2}+1\right]\,. \tag{2.5}$$

We take $m=1$ and take the non-relativistic massive limit, which corresponds to $f\longrightarrow 1$ (this means that we must scale the r and p by appropriate powers of the mass, giving p weight $1/2$ and giving r weight $-1/2$). We now see if we can extend the algebra. In this case, we can rewrite part of the envisioned Virasoro algebra as

$$[L_{-1},L_n]=\imath(n+1)L_{n-1}=\imath p\frac{\partial L_n}{\partial r} \tag{2.6}$$

$$[L_n,L_1]=-\imath(n-1)L_{n+1}=\imath r\frac{\partial L_n}{\partial p}\,, \tag{2.7}$$

where we evaluate the algebra classically (i.e. using Poisson brackets).[1] Using these two relations, we find

$$L_n=\frac{r^2}{n-2}\frac{\partial^2 L_{n-1}}{\partial r\partial p}+\frac{(n+1)r}{p}L_{n-1}\,. \tag{2.8}$$

[1] We use the convention $[r,p]=1$

Given that we have $L_0 = \frac{1}{2}rp$, we can use this recursive relation to see that we must have

$$L_n = -\frac{\imath}{2} r^{1+n} p^{1-n} . \tag{2.9}$$

Now, we must check that these L's satisfy the Virasoro algebra.

$$\begin{aligned}
[L_m, L_n] &= -\frac{1}{4}\left[r^{1+m}p^{1-m}, r^{1+n}p^{1-n}\right] \\
&= -\frac{1}{4}\Big(\big[(1+m)(1-n) - (1-m)(1+n)\big] r^{1+n+m} p^{1-(n+m)}\Big) \\
&= -\imath(m-n)L_{n+m}
\end{aligned} \tag{2.10}$$

We thus confirm that we in fact have the generators of the full Virasoro algebra. One should also note that the L's we have found do not seem to obey the usual relationship $L_n^\dagger = L_{-n}$.

3. The general case: scale-invariant hamiltonians

We now consider the most general scale invariant Hamiltonian we can write, given that p has dimension $1/2$ and r has dimension $-1/2$

$$H = \frac{p^2}{2f(u)} , \tag{3.1}$$

where f is an arbitrary function of $u = rp$. Motivated by our previous calculation and the results of [9], we make the following ansatz,

$$L_n = -\frac{\imath}{2} r^{1+n} p^{1-n} f^n . \tag{3.2}$$

Note that this reduces to the case in [9] and to the free particle case described above in the appropriate limits. Again, we must check that the full Virasoro algebra is satisfied.

$$\begin{aligned}
[L_m, L_n] &= -\frac{1}{4}\left[r^{1+m}p^{1-m}f^m, r^{1+n}p^{1-n}f^n\right] \\
&= -\imath(m-n)L_{m+n} - \frac{1}{4}\left[r^{2+n+m}p^{2-(n+m)}f^{n+m-1}\frac{\partial f}{\partial u}\right] \times \\
&\qquad \times\Big[(1+m)n - (1-m)n + (1-n)m - (1+n)m\Big] \\
&= -\imath(m-n)L_{m+n} .
\end{aligned} \tag{3.3}$$

This shows that given any scale invariant Hamiltonian of one variable, one can construct (classically) not only the generators of 0+1 dimensional conformal symmetry, but also the generators of the full Virasoro algebra.

4. The relativistic limit

We will now consider the massless relativistic limit. In this case, we do not scale r and p by powers of a mass. Thus, r has weight -1 and p has weight 1. We may rewrite this case in terms of our previous analysis by making the following change of variables

$$\tilde{r} = r^2\,, \qquad \tilde{p} = \frac{\partial L}{\partial \dot{\tilde{r}}} = p\frac{\partial \tilde{r}}{\partial r} = \frac{p}{2r}\,, \tag{4.1}$$

where we see, as expected, that $\tilde{r}$ has weight -1 and $\tilde{p}$ has weight 1 Note that $rp = \tilde{r}\tilde{p}/2$. If we now choose

$$f(rp) = rp\tilde{f}(\tilde{r}\tilde{p})\,, \tag{4.2}$$

then we find

$$L_n = -\frac{\imath}{2} r^{1+n} p^{1-n} f^n = -\frac{\imath}{2} r^{1+2n} p \tilde{f}^n = -\imath \tilde{r}^{1+n} \tilde{p} \tilde{f}^n\,. \tag{4.3}$$

Let us set $\tilde{f} = 1$. We see that $L_{-1} = -\imath\tilde{p}$ is then related to the relativistic Hamiltonian $H = \tilde{p}$ in the expected manner.[2] The charge-free subgroup provides a representation of the 0+1 conformal group as well [10].

One reason why we consider the relativisitic limit here is because it can easily be linked to the the representation of the Virasoro algebra in terms of diffeomorphisms of a circle. From that point of view, one may write the generators of the algebra as

$$L_n = \imath e^{\imath n\theta} \frac{\partial}{\partial \theta}\,. \tag{4.4}$$

Consider then the transformation

$$\tilde{r} = e^{\imath\theta}\,, \qquad L_n = \imath e^{\imath n\theta}\frac{\partial}{\partial\theta} = -\tilde{r}^{1+n}\frac{\partial}{\partial\tilde{r}} = -\imath\tilde{r}^{1+n}\tilde{p}\,. \tag{4.5}$$

Note, however, that the condition that $\tilde{r}$ be real amounts to the constraint that either $\theta = C$ or $\theta = \pi + C$ where C is purely imaginary. In some sense, we can think of the Virasoro algebra (in the relativistic free particle limit, at least) as arising from the diffeomorphisms of a circle, analytically continued to a real line.

5. Computing the central charge

The analysis of the previous section also provides insight into the quantization of the Virasoro algebra. The transition $-\imath\frac{\partial}{\partial\tilde{r}} \longrightarrow \tilde{p}$ is essentially the quantization of the

[2] We may instead consider the Hamiltonian $H = |p|$. This may be achieved this by setting $\tilde{f} = 1$ for positive p and $\tilde{f} = -1$ for negative p. Note that there are technical issues associated with the singular behavior at $p = 0$.

system. The transformation of the previous section then seems to give us an ordering for the quantum generators. In particular, we make the ansatz

$$L_n = -\imath \tilde{r}^{1+n} \tilde{p} \tag{5.1}$$

as a quantum operator. We then examine the full quantum algebra, and find

$$\begin{aligned}
[L_n, L_m] &= -\left[\tilde{r}^{1+n}\tilde{p}, \tilde{r}^{1+m}\tilde{p}\right] \\
&= -\tilde{r}^{1+n}\left[\tilde{p}, \tilde{r}^{1+m}\tilde{p}\right] - \left[\tilde{r}^{1+n}, \tilde{r}^{1+m}\tilde{p}\right]\tilde{p} \\
&= -\tilde{r}^{1+n}\left[\tilde{p}, \tilde{r}^{1+m}\right]\tilde{p} - \tilde{r}^{1+m}\left[\tilde{r}^{1+n}, \tilde{p}\right]\tilde{p} \\
&= -\imath(n-m)\tilde{r}^{1+n+m}\tilde{p} = (n-m)L_{n+m}\,.
\end{aligned} \tag{5.2}$$

In the case of the free relativistic particle, we have exhibited generators which satisfy the full quantum Virasoro algebra with no central charge.

6. Multiplets of states

One might expect to be able to find the states organized into highest weight representations of the Virasoro algebra. In fact, this is somewhat difficult. In 1+1 conformal field theory, L_0 is the Hamiltonian and is bounded from below. Since all of the other L's will shift the eigenvalue of L_0, one finds that the eigenstate of L_0 corresponding to its minimum eigenvalue must be annihilated by the half of the L's which are lowering operators. It is upon this eigenstate that one builds multiplets of the algebra. In our case, however, L_0 is the generator of dilatations, and its eigenvalues are not bounded. Instead, the Hamiltonian is given (up to constant factors) by L_{-1}. In order to find an equivalent construction, we must find linear combinations of the L's such that

$$\left[H, \sum_n a_n L_n\right] = C \sum_n a_n L_n\,. \tag{6.1}$$

These operators will then be the raising and lowering operators of the representation. Using the commutation relations, one can easily see that the raising and lowering operators will be of the form

$$e^{\imath k r} p\,. \tag{6.2}$$

One could see this just as well by remembering that since $H = p$, the raising and lowering operators are just operators which shift the momentum. The only state which is annihilated by these operators is the zero momentum state. But since it is annihilated by all of these operators, one cannot build the multiplet upon it. In that sense, there is no highest weight representation.

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7. Hermiticity and global time

We have noted that, for the constructions provided in the previous sections, one does not have the condition $L_n^\dagger = L_{-n}$. This seems unusual given that, in some cases, the scale-invariant mechanics actually describes the motion of a particle in the background of AdS_2. The L_n's of the bulk are the Fourier modes of the bulk stress-energy tensor, and as such must obey the hermiticity condition. One might ask why the same does not appear to be true for the particle. The answer is related to the choice of time.

Both the free non-relativistic particle and free relativistic particle are limits of the Hamiltonian found in [9]. That Hamiltonian was derived from the Born-Infeld action for the radial excitations of a particle in the background of $AdS_2 \times S^2$ given in Poincaré coordinates. These coordinates do not cover the entire anti-de Sitter space. One can write these coordinates in such a way that the space they cover is a half-plane which is conformally flat.

$$ds^2 = \frac{1}{u^2}(-dt^2 + du^2)\,, \qquad u > 0\,. \tag{7.1}$$

If one thinks of these coordinates as parametizing the upper half of the complex plane, then time translation is the same as translation along the real axis. Since the L_n's generate transformations of the upper half-plane, one can see that L_{-1} is the generator of translation along the real axis. Therefore, it is reasonable to associate the Hamiltonian with L_{-1}. One then easily sees that one can identify $L_{-1,0,1}$ with H,D and K in such a way that the Virasoro algebra and 0+1 conformal algebra both close.

But this is not the usual situation with which we are familiar from 1+1 conformal field theory. In that case, we generally find that the Hamiltonian is given by L_0, and that $L_n^\dagger = L_{-n}$. In order to find similar relations, one must choose the time coordinate such that time translation is generated by L_0. Global time coordinates [3] are well suited for this purpose. These coordinates may be written as

$$ds^2 = -(1 + a^2 r^2)dt^2 + (1 + a^2 r^2)^{-1} dr^2\,. \tag{7.2}$$

After the transformation

$$\tan(\sigma) = ar \qquad \rho = e^{at} \tag{7.3}$$

and a Wick rotation, one finds the metric

$$ds^2 = \frac{1}{a^2\rho^2 cos^2(\sigma)}(d\rho^2 + \rho^2 d\sigma^2)\,, \qquad -\frac{\pi}{2} < \sigma < \frac{\pi}{2}\,. \tag{7.4}$$

It is clear that time translation now corresponds to scaling of the half-plane, which is generated by L_0.

[3] We will actually parameterize $CAdS_2$, the covering space of AdS_2.

Let us now write the action of a particle in an anti-de Sitter space background given in global time coordinates.

$$L = -m\sqrt{(1+a^2r^2) - (1+a^2r^2)^{-1}\dot{r}^2}\,. \tag{7.5}$$

The Hamiltonian associated with this action is

$$H = \left(1+a^2r^2\right)^{\frac{1}{2}}\left(m^2+p^2+a^2r^2p^2\right)^{\frac{1}{2}}\,. \tag{7.6}$$

In the large radius, relativisitic limit the Hamiltonian reduces to $H = p$ (as one would expect for a relativisitic particle in nearly flat space). Following the logic used above, we set $L_0 = H = p$.

We must now attempt to find the generators of the rest of the Virasoro algebra. Note that we should not expect $L_{-1,0,1}$ to be expressed in terms of H, D and K (which is the formulation appropriate to Poincaré coordinates). If $L_0 = H$, there is no way to write L_1 or L_{-1} in terms of H, K or D such that they close under the Virasoro algebra. But one quickly sees that in order for $[L_m, L_0] = -\imath m L_m$ to hold (classically), L_m must be of the form

$$L_m = g_m(p)e^{-\imath m r}\,. \tag{7.7}$$

We make the ansatz

$$L_m = e^{-\imath m r}p \tag{7.8}$$

and can readily verify that this choice satisfies the classical Virasoro algebra under Poisson brackets. We also note that $L_n^\dagger = L_{-n}$ (classically). This is exactly what we hoped to find in order to match our intuition from 1+1 CFT.

The next question is whether or not the above system can be quantized. Consider the ansatz

$$L_m = e^{-\frac{\imath m}{2}r}pe^{-\frac{\imath m}{2}r}\,, \tag{7.9}$$

where L_m is now a quantum operator. It is clear that $L_m^\dagger = L_{-m}$. One also sees that

$$\begin{aligned}
[L_n, L_m] &= e^{-\frac{\imath n}{2}r}\left[p, e^{-\frac{\imath m}{2}r}\right]pe^{-\frac{\imath m}{2}r}e^{-\frac{\imath n}{2}r} + e^{-\frac{\imath n}{2}r}e^{-\frac{\imath m}{2}r}p\left[p, e^{-\frac{\imath m}{2}r}\right]e^{-\frac{\imath n}{2}r} + \\
&\quad + e^{-\frac{\imath n}{2}r}pe^{-\frac{\imath m}{2}r}\left[e^{-\frac{\imath n}{2}r}, p\right]e^{-\frac{\imath m}{2}r} + e^{-\frac{\imath m}{2}r}\left[e^{-\frac{\imath n}{2}r}, p\right]e^{-\frac{\imath m}{2}r}pe^{-\frac{\imath n}{2}r} \\
&= (n-m)L_{n+m} + \frac{n}{2}e^{-\frac{\imath n}{2}r}\left[p, e^{-\frac{\imath m}{2}r}\right]e^{-\frac{\imath n}{2}r}e^{-\frac{\imath m}{2}r} + \frac{n}{2}e^{-\frac{\imath m}{2}r}e^{-\frac{\imath n}{2}r}\left[e^{-\frac{\imath m}{2}r}, p\right]e^{-\frac{\imath n}{2}r} \\
&= (n-m)L_{n+m}\,.
\end{aligned} \tag{7.10}$$

Therefore we can conclude that for the free relativistic particle in global time, the Virasoro algebra satisfies hermiticity and has no central charge.

One may also consider the limit of small AdS radius and small m. In that case, we find

$$L_0 = H = pr^2\,. \tag{7.11}$$

Take the ansatz

$$L_m = pr^2 e^{\frac{\imath m}{r}} . \tag{7.12}$$

Again, we can verify that the Virasoro algebra does indeed close under Poisson brackets. To quantize this system, we make the ansatz

$$L_m = e^{\frac{\imath m}{2r}} rpr e^{\frac{\imath m}{2r}} \tag{7.13}$$

as a quantum operator. We then find that the quantum Virasoro algebra closes with no central charge.

8. Conclusion

It is interesting to note that the AdS/CFT correspondence served only as a motivational tool in the above derivation. The conformal mechanics described above need not be related to a theory on anti-de Sitter space. It is worthwhile to further investigate the relationship between conformal mechanics and gravity theories on AdS_2 [11, 12].

Additionally, the arguments given above only apply to the mechanics of a single variable. It would be interesting to conjecture that any conformal mechanics of any arbitrary number of degrees of freedom contains the symmetries of the Virasoro algebra. If true, this would more closely cement the relationship found in [4] between conformal quantum mechanics and 1+1 conformal field theory.

Finally, it would very interesting to completely understand the quantization of these conformal mechanics models. More concretely, one would like to know if there is a way to determine the central charge and normal-ordering conventions for arbitary scale-invariant Hamiltonians. Thus far, all of the systems which we have been able to quantize could be thought of as describing a limit of a particle in the background of AdS_2. Since the bulk theory is a two-dimensional quantum gravity theory, the central charge of the bulk Virasoro algebra should vanish (when all ghosts are included). One might speculate that this is the reason why the central charge of the quantum mechanics has vanished in all cases we have found. One would like to know if a non-vanishing central charge could perhaps be found for quantized systems which did not represent a particle in the background of AdS_2.

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